Quaternion CR-Submanifolds of a Locally Conformal Quaternion Kaehler Manifold

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Dedicated to the memory of Dr. Ümran Pekmen

Abstract. *The purpose of the present paper is to study the differential geometric properties of a quaternion CR-submanifold in a locally conformal quaternion Kaehler manifold.*

1 Introduction

The concept of the locally conformal Kaehler manifolds was introduced by I.Vaisman in [19]. Since then many papers appeared on these manifolds and their submanifolds (See: [8] and its references). However,the geometry of the locally conformal quaternion Kaehler manifolds has been studied in the last ten years, [8], [11],[12], [13] ,[14], [15], [16] and their QR-Submanifolds have been studied in [17]

A locally conformal quaternion Kaehler manifold (Shortly, l.c.q.K manifold) is a quaternion Hermitian manifold whose metric is conformal to a quaternion Kaehler metric in some neighborhood of each point. The main difference between locally conformal Kaehler manifold and l.c.q.K. manifold is that the Lee form of a compact l.c.q.K. manifold can be choosen as parallel form without any restrictions[8]. It is known that this property is not guaranteed in the complex case,[18].

On the other hand,A.Bejancu [2] defined and studied CR-submanifolds of a Kaehler manifold. Since then many papers appeared on this topic, [2],[4],[6],[7]. Moreover, Barros, Chen and Urbano defined quaternion CR-submanifold of a quaternion Kaehler manifold as analogy with CR-submanifold of a Kaehler manifold [1].

In this paper, we study the geometry of quaternion CR-submanifolds of a l.c.q.K. manifold.In section 2, we give some basic definitions, formulae and result

which will be used in this paper. In section 3, we study the geometry of quaternion CR-submanifolds of a l.c.q.K. manifold. In this section, we investigate the geometry of leaves. In section 4, we consider totally umbilical quaternion CR-submanifolds. Finally, section 5 is devoted to investigate the topology of a quaternion CR-submanifold.

2 Locally Conformal Quaternion Kaehler Manifolds

We denote a quaternion Hermitian manifold by $\left(\bar{M}, g, H\right)$, where H is a subbundle of $End(T\bar{M})$ of rank 3 which is spanned by almost complex structures J_1, J_2, and J_3. We recall that a quaternion Hermitian metric g is said to be a quaternion Kaehler metric if its Levi-Civita connection $\bar{\nabla}$ satisfies $\bar{\nabla}H \subset H$.

A quaternion Hermitian manifold with metric g is a l.c.q.K. manifold if over neighborhoods $\{U_i\}$ covering M, $g|_{U_i} = e^{f_i}g_i'$ with g_i' a quaternion Kaehler metric on U_i. In this case, the Lee form ω is locally defined by $\omega|_{U_i} = df_i$ and satisfies

$$d\Theta = \omega \wedge \Theta, d\omega = 0 \tag{2.1}$$

where $\Theta = \sum_{\alpha=1}^{3} \Omega_\alpha \wedge \Omega_\alpha$ is the Kaehler $4-$form. We note that property (2.1) is also a sufficient condition for a quaternion Hermitian metric to be a l.c.q.K. metric [11].

Let $\bar{\nabla}$ be the Levi-Civita connection of g. We define a Lee vector field B by $\omega(X) = g(X, B)$ on $\bar{M}$. We have another torsionless linear connection $\bar{D}$ called the Weyl connection. The Weyl connection $\bar{D}$ related to the Levi-Civita connection $\bar{\nabla}$ of g by the formula

$$\bar{D}_X Y = \bar{\nabla}_X Y - \frac{1}{2}\left\{\omega\left(X\right)Y + \omega\left(Y\right)X - g(X,Y)B\right\} \tag{2.2}$$

for any $X, Y \in \Gamma\left(T\bar{M}\right)$, where $B = \omega^{\#}$ is the Lee vector field [8].

Let $\bar{M}$ be a l.c.q.K. manifold and $\bar{\nabla}$ be the connection of $\bar{M}$. Then the Weyl connection does not preserve the compatible almost complex structures individually but only their 3-dimensional bundle H, that is,

$$\bar{D}J_a = \sum Q_{ab} \otimes J_b \tag{2.3}$$

for $a, b = 1, 2, 3$, and Q_{ab} is a skew-symmetric matrix of local forms [13]. Thus, from (2.1) and (2.2) we have

$$\begin{aligned}\bar{\nabla}_X J_a Y &= J_a\bar{\nabla}_X Y + \tfrac{1}{2}\left\{\theta_o\left(Y\right)X\omega\left(Y\right)J_a X - \Omega left(X,Y)B\right. \\ &\left. + g(X,Y)J_a B\right\} + Q_{ab}(X)J_b Y + Q_{ac}(X)J_c Y\end{aligned} \tag{2.4}$$

for any $X, Y \in \Gamma\left(T\bar{M}\right)$, where $\theta_o = \omega o J_a$.

Let $R^{\bar{D}}$ and $\bar{R}$ be the curvature tensor fields of the connections $\bar{D}$ and $\bar{\nabla}$, respectively. Then we have

$$
\begin{aligned}
R^{\bar{D}}(X,Y)Z \quad &= \bar{R}(X,Y)Z - \tfrac{1}{2}\{[(\bar{\nabla}_X\omega)Z + \tfrac{1}{2}\omega(X)\omega(Z)]Y \\
&\quad [(\bar{\nabla}_Y\omega)Z + \tfrac{1}{2}\omega(Y)\omega(Z)]X + ((\bar{\nabla}_X\omega)Y)Z \\
&\quad -((\bar{\nabla}_Y\omega)X)Z - g(Y,Z)(\bar{\nabla}_X B + \tfrac{1}{2}\omega(X)B) \\
&\quad +g(X,Z)(\bar{\nabla}_Y B + \tfrac{1}{2}\omega(Y)B\} \\
&\quad -\tfrac{1}{4}\|\omega\|^2(g(Y,Z)X - g(X,Z)Y)
\end{aligned}
\tag{2.5}
$$

for any $X,Y \in \Gamma(T\bar{M})$

Next we give the following theorem which will be useful later.

Theorem 2.1 *[12] Let $(\bar{M},\bar{g},H)$ be a compact quaternion Hermitian Weyl manifold, non-quaternion Kaehler, whose foliation F has compact leaves. Then the leaves space $P = \bar{M}/F$ is a compact quaternion Kaehler orbifold with positive scalar curvature, the projection is a Riemannian, totally geodesic submersion and a fibre bundle map with fibres as described in proposition 4.10 of [12], where F is locally generated by $B, J_1B = B_1, B_2, B_3$.*

If F is a regular foliation, then $P = \bar{M}/F$ is a compact quaternion Kaehler manifold [10].

Let $\bar{M}$ be a l.c.q.K. manifold and M be any submanifold of $\bar{M}$. The formulae of Gauss and Weingarten are given by

$$
\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)
\tag{2.6}
$$

and

$$
\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V
\tag{2.7}
$$

for vector fields X,Y tangent to M and any vector field V normal to M, where ∇ is the induced Riemann connection in M, h is the second fundamental form, A_V is the fundamental tensor field of Weingarten with respect to the normal section V and $\nabla^\perp$ is the normal connection. Moreover, we have the relation

$$
g(h(X,Y),V) = g(A_V X, Y).
\tag{2.8}
$$

The equations of Gauss and Codazzi are given respectively by [5]

$$
\begin{aligned}
R(X,Y;Z,W) \quad &= \bar{R}(X,Y;Z,W) + g(h(X,W),h(Y,Z)) \\
&\quad -g(h(X,Z),h(Y,W))
\end{aligned}
\tag{2.9}
$$

and

$$
(\bar{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z)
\tag{2.10}
$$

for $X,Y,Z \in \Gamma(TM)$, where $\bar{R}, R$ is the curvature tensor corresponding to the connection $\bar{\nabla}, \nabla$ respectively and $\perp$ in (2.10) denotes the normal component. The covariant derivative $(\bar{\nabla}_X h)(Y,Z)$ is given by

$$
(\bar{\nabla}_X h)(Y,Z)) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
$$

3 Quaternion CR-Submanifolds of a l.c.q.K. manifold

First, we give definition of a quaternion CR-submanifold of a l.c.q.K. manifold as the definition of quaternion CR-submanifolds of a quaternion Kaehler manifold.

Definition 3.1 *A submanifold M of a l.c.q.K. manifold $\bar{M}$ is called a quaternion CR-submanifold if there exists two orthogonal complementary distributions D and $D^\perp$ such that D is invariant under J_a, i.e, $J_a D \subseteq D$, $a = 1, 2, 3$. and $D^\perp$ is totally real, i.e. $J_a D^\perp \subseteq TM^\perp$, $a = 1, 2, 3$.*

A submanifold M of a l.c.q.K. manifold $\bar{M}$ is called a quaternion submanifold(resp. totally real submanifold) if $dim D^\perp = 0$ (resp. dim$D$=0). A quaternion CR-submanifold is called proper quaternion CR-submanifold if it is neither quaternion nor totally real.

By the definition a quaternion CR-submanifold, we have

$$TM = D \oplus D^\perp \tag{3.1}$$

and

$$TM^\perp = J_a D^\perp \oplus \mu \tag{3.2}$$

where μ is orthogonal complement of $J_a D^\perp$ in the normal bundle is invariant subbundle of $\Gamma(TM^\perp)$ under J_a.

Now, let M be a quaternion CR-submanifold of a l.c.q.K. manifold $\bar{M}$. For each vector field X tangent to M we put

$$J_a X = \phi_a X + \varpi_a X \tag{3.3}$$

where $\phi_a X \in \Gamma(D)$ and $\varpi_a X \in \Gamma(J_a D^\perp)$. Also, for each vector field V normal to M we put

$$J_a V = f_a V + t_a V \tag{3.4}$$

where $f_a V \in \Gamma(D^\perp)$ and $t_a V \in \Gamma(\mu)$.

Now, we will give an example for quaternion CR-submanifolds of a l.c.q.K. manifold.

Example 3.1 *Let $\bar{M}$ be a l.c.q.K. manifold. Assume that the foliation F is regular. Then $P = \bar{M}/F$ is a compact quaternion Kaehler manifold (cf:Theorem.2.1). We denote almost complex structures of $\bar{M}$ and P by J_a and J'_a, respectively. Now we consider the following commutative diagram:*

$$
\begin{array}{ccc}
\bar{M} & \xrightarrow{\pi} & P = \bar{M}/F \\
\uparrow i & & \uparrow j \\
N & \xrightarrow{\bar{\pi}} & \bar{N}
\end{array}
$$

where N and $\bar{N}$ are submanifolds of $\bar{M}$ and P, respectively. We denote the horizontal lift by $*$. Then we have

$$(J'_a X)^* = J_a X^*. \tag{3.5}$$

We note that the projection π is a totally geodesic Riemannian submersion and a fibre bundle map. Hence $\bar{\pi}$ is also a Riemannian submersion. We denote the vertical distribution of the Riemannian submersion π by υ, i.e. $\ker\pi_* = \upsilon$. Let $\bar{H}$ be the horizontal distribution of π. Then we have $T\bar{M} = \bar{H} \oplus \upsilon$. We denote the horizontal distribution of $\bar{\pi}$ by H_0. We will investigate the relation between normal spaces of N and $\bar{N}$. We denote the Riemannian metrics of $\bar{M}$ and P by g and g', respectively. Let V^* be the horizontal lift of $V \in \Gamma(T\bar{N}^\perp)$. Then we get

$$\begin{aligned}
g(V^*, X) &= g((\pi_*)^* V, X) \\
&= g'(\pi_* X, V) \\
&= 0,
\end{aligned}$$

for any $X \in H_0$. Thus, $(T\bar{N}^\perp)^*$ is orthogonal to H_0. Note that the normal space is always horizontal. Hence $(T\bar{N}^\perp)^*$ is orthogonal to υ. Consequently, we have $(T\bar{N}^\perp)^* \subseteq TN^\perp$. Since π is a Riemannian submersion we get

$$(T\bar{N}^\perp)^* = TN^\perp. \tag{3.6}$$

Now, let ϕ_a and ϖ_a be the operators on $\bar{N}$ appearing in (3.3). We denote the operators in N corresponding to ϕ_a and ϖ_a by ϕ'_a and ϖ'_a, respectively. From (3.5) and (3.6) we obtain

$$(\phi_a X)^* = \phi'_a X^* \tag{3.7}$$

and

$$(\varpi_a X)^* = \varpi'_a X^*. \tag{3.8}$$

So, from (3.7) and (3.8) we see that N is a quaternion CR-submanifold of $\bar{M}$ if and only if $\bar{N}$ is a quaternion CR-submanifold of P.

In the rest of this section, we will investigate the geometry of leaves on quaternion CR-submanifolds.

In [1], Barros, Chen and Urbano showed that the anti-invariant distribution $D^\perp$ of a quaternion Kaehler manifold is integrable. In the next theorem, we will see that is still true for a quaternion CR-submanifold of a l.c.q.K. manifold.

Theorem 3.1 *Let M be a proper quaternion CR-submanifold of a l.c.q.K. manifold. Then the anti-invariant distribution $D^\perp$ of M is integrable.*

Proof. From (2.4),(2.6) and (2.7) we obtain

$$-g(A_{J_a W} T, X) = g(\nabla_T W, X) + g(T, W) g(J_a B, X) \tag{3.9}$$

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for any $X \in \Gamma(D)$ and $T, W \in \Gamma(D^\perp)$.Interchanging T and W in (3.9) and subtracting we get

$$g(A_{J_aT}W - A_{J_aW}T, X) = g(J_a[T,W], X). \qquad (3.10)$$

On the other hand, since $\bar{\nabla}$ is a metric connection and A is self-adjoint we obtain

$$g(A_{J_aT}W, X) = -g(W, \bar{\nabla}_X J_a T). \qquad (3.11)$$

In this equation, using (2.4) and (2.6) we have

$$g(A_{J_aT}W, X) = g(A_{J_aW}T, X). \qquad (3.12)$$

Thus, from (3.10) and (3.12), we obtain

$$g([T,W], J_a X) = 0,$$

which proves our assertion.

Lemma 3.1 *Let M be a quaternion CR-submanifold of a l.c.q.K. manifold. Then quaternion distribution D is minimal if and only if the Lee vector field is orthogonal to the anti-invariant distribution $D^\perp$.*

Proof. Since $\bar{\nabla}$ is a metric connection, from (2.4) and (2.7) we obtain

$$g(\nabla_X X, Z) = g(A_{J_aZ}X, J_aX) - \frac{1}{2}\|X\|^2 \omega(Z) \qquad (3.13)$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D)$. In a similar way we have

$$g(\nabla_{J_aX} J_aX, Z) = -g(A_{J_aZ}X, J_aX) - \frac{1}{2}\|X\|^2 \omega(Z) \qquad (3.14)$$

Thus from (3.13) and (3.14) we have

$$g(\nabla_X X, Z) + g(\nabla_{J_aX} J_aX, Z) = 0 \Leftrightarrow \omega(Z) = 0$$

Now, we will discuss the integrability of the quaternion distribution. First we give a lemma.

Lemma 3.2 *Let M be a quaternion CR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then we have*

$$h(X, J_aY) = \varpi_a \nabla_X Y + t_a h(X,Y) + \frac{1}{2}\{g(X,Y)(J_aB)^\perp - \Omega(X,Y)B^\perp\} \quad (3.15)$$

and

$$h(X, J_aY) - h(Y, J_aX) = \varpi([X,Y]) + \Omega(X,Y)B^\perp \qquad (3.16)$$

for any $X, Y \in \Gamma(D)$, where $B^\perp = norB$.

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Proof. Using (2.4), (2.6), (2.7) and a comparison between normal components we derive the (3.15). Then (3.16) is direct consequence of (3.15).

Definition 3.2 *Let M be a quaternion CR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then M is called D-geodesic if $h(X,Y) = 0$ for $X, Y \in \Gamma(D)$.*

From Lemma 3.2 and Definition.3.2 we have:

Theorem 3.2 *Let M be a quaternion CR-submanifold of a l.cq.K. manifold $\bar{M}$. Assume that the Lee vector field is tangent to M. Then the following assertions are equivalent:*

1. for any $X, Y \in \Gamma(D)$

$$h(X, J_a Y) = h(Y, J_a X).$$

2. M is D-geodesic.

3. D is integrable.

The proof is similar to that of Theorem 2.1 in [3] (Also, Theorem 3.2 in [17]). So we omit it here.

From Lemma 3.2 we have the following corollary.

Corollary 3.1 *Let M be a quaternion CR-submanifold of a l.cq.K. manifold $\bar{M}$. If M is D-geodesic and the Lee vector field orthogonal to $D^{\perp}$, then each leaf of D is totally geodesic.*

Corollary 3.2 *Let M be a quaternion CR-submanifold of a l.cq.K. manifold $\bar{M}$. If $h(X, Z) \subset \mu$ for $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$ and the Lee vector field is orthogonal to D, then each leaf of $D^{\perp}$ is totally geodesic in M.*

Proof. From (2.4), (2.6) and (2.7) we obtain

$$-g(A_{J_a W} Z, X) = -g(\nabla_Z W, J_a X) + \frac{1}{2} g(Z, W) g(J_a B, X)$$

for any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$. Using (2.8) we arrive at

$$g(h(Z, X), J_a W) = g(\nabla_Z W, J_a X) + \frac{1}{2} g(Z, W) g(B, J_a X)$$

which proves our assertion.

4 Umbilical Quaternion CR-Submanifolds of l.c.q.K. Manifolds

Let $\bar{M}$ be a compact l.c.q.K. manifold. Then we can choose the fixed metric g such that

    i) The fixed metric g makes ω parallel, i.e.

$$\bar{\nabla}\omega = 0 \qquad (4.1)$$

ii)

$$\|\omega\|^2 = 1 \qquad (4.2)$$

(See:[11]). From now on we will denote a compact l.c.q.K. manifold by $\bar{M}$ in this section.

Let M be a quaternion CR-submanifold of l.c.q.K. manifold $\bar{M}$. Then M is called totally umbilical if we have

$$h(X,Y) = g(X,Y)H \qquad (4.3)$$

for any X, Y tangent to M, where H is the mean curvature vector field defined by $H = \frac{1}{m}Trace(h)$. We say that M is totally geodesic if $h = 0$ identically on M.

Theorem 4.1 *Let M be a quaternion CR-submanifold of a l.c.q.K. manifold If the Lee vector field B is tangent to M, then we have*

$$K_{\bar{M}}(X,Y) \leq \frac{1}{4} \qquad (4.4)$$

for any orthonormal vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. The equality (4.4) holds if and only if $\bar{M}$ is a quaternion Kaehler manifold

Proof. Let $R^{\bar{D}}$ be the curvature tensor field of the Weyl connection $\bar{D}$. Then we have

$$R^{\bar{D}}(X,Y)J_a Z - J_a R^{\bar{D}}(X,Y)Z = \alpha(X,Y)J_b Z - \beta(X,Y)J_c Z \qquad (4.5)$$

where

$$\alpha = dQ_{ab} + Q_{cb} \wedge Q_{ac}$$

and

$$\beta = dQ_{ac} + Q_{bc} \wedge Q_{ab}.$$

Taking account (2.5), (4.1), (4.2) and (4.5) we obtain

$$0 \;\; = -\bar{R}(X,Y,X,Y) + \bar{R}(X,Y,J_a X, J_a Y) \atop \tfrac{1}{4}\omega(X)\omega(X) + \tfrac{1}{4}\omega(Y)\omega(Y) - \tfrac{1}{4} \qquad (4.6)$$

for any orthonormal vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. On the other hand, from (4.3) and Codazzi equation we have

$$\bar{R}(X,Y,Z,V) = g(Y,Z)g(\nabla_X^\perp H, V) - g(X,Z)g(\nabla_Y^\perp H, V) \qquad (4.7)$$

for any vector fields X, Y, Z tangent to M and V normal to M. Thus using (4.7) we get

$$\bar{R}(X, Y, J_a X, J_a Y) = 0 \qquad (4.8)$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Using (4.6) and (4.8) we arrive at

$$\bar{R}(X, Y, Y, X) = -\frac{1}{4}\omega(X)^2 - \frac{1}{4}\omega(Y)^2 + \frac{1}{4}. \qquad (4.9)$$

If the Lee vector field is tangent to M from (4.9) we have (4.4). In view of (4.9) the equality case of (4.4) is valid if and only if

$$\omega(X)^2 + \omega(Y)^2 = 0$$

for any orthonormal vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Thus we obtain

$$\omega(X) = 0 \qquad (4.10)$$

and

$$\omega(Y) = 0. \qquad (4.11)$$

From (4.10) and (4.11) we obtain $B \in \Gamma(D)$ and $B \in \Gamma(D^\perp)$, respectively. Since $D \cap D^\perp = \{0\}$ we have $B = 0$.

Now we give an another theorem for totally umbilical proper quaternion CR-submanifolds of a l.c.q.K. manifold. We start with the following preparatory result.

Lemma 4.1 *Let M be a totally umbilical quaternion CR-submanifold of a l.c.q.K. manifold. Assume that the Lee vector field is tangent to M. Then we have*

$$H \in \Gamma(J_a D^\perp) \qquad (4.12)$$

Proof. Since B is tangent to M, from Lemma 3.2 we obtain

$$g(h(X, J_a Y), N) = -g(h(X, Y), J_a N)$$

for any $N \in \Gamma(\mu)$ and $X, Y \in \Gamma(D)$. Since M is totally umbilical we get

$$g(X, J_a Y)g(H, N) = -g(X, Y)g(H, J_a N).$$

Thus for $X = J_a Y$ we have $g(H, N) = 0$, hence we obtain $H \in \Gamma(J_a D^\perp)$

Theorem 4.2 *Let M be a totally umbilical proper quaternion CR-submanifold of a l.c.q.K. manifold. Assume that the Lee vector field is tangent to M. Then*

1. *M is totally geodesic or*

2. *the totally real distribution is one dimensional.*

Proof. We take $Z, W \in \Gamma(D)$ and using totally umbilicalness of M together (2.4), (2.6) and (2.7) we have

$$-A_{J_aW}Z = \phi_a \nabla_Z W + g(Z,W)J_a H + \frac{1}{2}\theta(W)Z + \frac{1}{2}g(Z,W)(J_a B)^T.$$

Taking inner product with Z in $D^\perp$ it follows that

$$-g(A_{J_aW}Z, Z) = -g(Z,W)g(H, J_a Z) + \frac{1}{2}\theta(W)g(Z,Z) - \frac{1}{2}g(Z,W)g(B, J_a Z)$$

Since B is tangent to M, from (2.8) and (4.3) we have

$$g(Z,Z)g(H, J_a W) = \frac{1}{2}g(Z,W)g(H, J_a Z). \tag{4.13}$$

Interchanging Z and W in (4.13) we obtain

$$g(W,W)g(H, J_a Z) = \frac{1}{2}g(Z,W)g(H, J_a W). \tag{4.14}$$

From (4.13) and (4.14), one can immediately get

$$g(H, J_a W) = \frac{g(Z,W)^2}{\parallel Z \parallel^2 \parallel W \parallel^2}g(H, J_a W). \tag{4.15}$$

From Lemma 4.1., the possible solutions of (4.15) are
a) $H = 0$ b)$Z//W$
Suppose condition a) holds,i.e., $H = 0$ which implies that M is totally geodesic. If b) is satisfies in (4.15) then $\dim D^\perp = 1$ which implies that the totally real distribution is one dimensional.

5 Cohomology of Quaternion CR-Submanifolds

Assume that M be a quaternion CR-submanifold of $4n$ dimensional l.c.q.K. manifold $\bar{M}$. Let $p = dim_Q D, q = dim D^\perp$ Then we choose an orthonormal frame $\{e_1, ..., e_p,\ e_{p+1} = J_1 e_1, ..., e_{4p},\ E_1, ..., E_q,\ J_1 E_1, ..., J_1 E_q,\ J_2 E_1, ..., J_2 E_q, J_3 E_1, ..., J_3 E_q,\ V_1, ..., V_r,\ V_{r+1} = J_1 V_1, ..., V_{4r}\}$ in $\bar{M}$ such that restricted to $M, \{e_1, ..., e_p, e_{p+1} = J_1 e_1, ..., e_{4p}\}$ are in D and $\{E_1, ..., E_q\}$ are in $D^\perp$. We denote by $\{w^1, ..., w^4 p\}$ the 1-forms on M satisfying

$$w^i(Z) = 0 \quad w^i(e_j) = \delta_{ij} \quad i,j = 1, ..., 4p \tag{5.1}$$

for any $Z \in \Gamma(D^\perp)$, where $e_{p+j} = J_1 e_j, e_{2p+j} = j J_2 e_j, e_{3p+j} = J_3 e_j$Then

$$w = w^1 \wedge ... \wedge w^4 p \tag{5.2}$$

defines a $4p$-form on M. From (5.2) we obtain

$$dw = \sum_{i=1}^{4p}(-1)^i w^1 \wedge ... \wedge ... \wedge w^{4p} \tag{5.3}$$

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Thus from (5.1) and (5.3) we obtain that $dw = 0$ if and only if

$$dw(Z_1, Z_2, X_1, ..., X_{4p-1}) = 0 \tag{5.4}$$

and

$$dw(Z_1, X_1, ..., X_{4p}) = 0 \tag{5.5}$$

for any $Z_1, Z_2 \in \Gamma(D^\perp)$ and $X_1, ..., X_{4p} \in \Gamma(D)$. We see that (5.4) holds if and only if $D^\perp$ is integrable and (5.5) holds if and only if D is minimal.Thus from Theorem.3.1 and Lemma.3.1, we have the following theorem.

Theorem 5.1 *Let M be a closed quaternion CR-submanifold of a l.c.q.K. manifold. If the Lee vector field is orthogonal to the anti-invariant distribution $D^\perp$ then the 4p-form w defines a canonical de Rham cohomology class [w] in $H^{4p}(M, R)$*

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